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Turbulent erosion of a stably stratified fluid as a test of intermittency models

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As a model for the erosion of a stably stratified fluid by an overlying turbulent region, the rate of advance of the mixing interface is calculated in terms of the probability distribution for eddies as a function of their size and velocity. Predicted rates of advance are then evaluated for two current models of intermittency in small-scale turbulence. Compared to available experimental data, one intermittency model, the β-model, is found to be in good agreement, while the other, the log-normal hypothesis, is discordant; unfortunately, the Reynolds number of the existing experiments is not large enough to yield a definitive test. At higher *Re*, similar experiments might be a useful complement to the alternative of measuring high-order moments. Some related experiments are suggested to test the model and measure the fractal dimension D.

1. Introduction

It is not unusual to have a fluid at rest with a stable density stratification in the vertical direction, and with the variation of density due to a miscible contaminant that obeys an advective diffusion equation ('salt', or 'heat in Boussinesq approximation'). The oceans are, of course, stably stratified in just this way, by both increasing salinity with depth and by decreasing temperature (for recent introductory reviews, see Flatte *et al.* 1979, ch. 1; Garrett & Munk 1979). If, overlying the stably stratified region, one has a mixed region (also like the oceans) in which the fluid is in driven, turbulent motion, then there is a continual erosion of the stable fluid as wisps of it are entrained, and ultimately mixed, into the turbulent flow. In the ocean, the phenomenon is called 'erosion of the thermocline', and it is investigated both by observation *in situ* (e.g. Grant, Moilliet & Vogel 1968), by laboratory experiment (e.g. Rouse & Dodu 1955; Turner 1968, 1973; Thompson 1969; Kato & Phillips 1969; Kantha & Phillips 1977), and by theoretical modelling (e.g. Pollard, Rhines & Thompson 1973; Niiler 1975). Phillips (1977, § 6.7) reviews the subject.

It does not appear to have been pointed out previously that the body of experimental data on this subject, turbulent erosion of a stably stratified fluid, taken together with a straightforward theoretical model, has direct bearing on the apparently rather different theoretical problem of producing a satisfactory model for small-scale intermittency in turbulent flow. The purpose of this paper is to indicate a connection between these two areas of current interest. It turns out that of two current models for small-scale intermittency, both of which generalize the original Kolmogorov (1941) self-similar theory, namely the Kolmogorov (1962) 'log-normal hypothesis' and the 'β-model' of Frisch, Sulem & Nelkin (1978), one agrees remarkably well with the existing data on turbulent erosion, while the other seems to be in disagreement.

Unfortunately, the Reynolds number of the experiments is not sufficiently large to allow any definitive conclusion. At larger *Re*, testing models of intermittency against macroscopic erosion rates might be a complimentary, and perhaps therefore useful, alternative to testing them against measured high-order statistical moments of turbulent flows.

In § 2, a model is developed which relates the intermittency, as a function of scale and velocity, to the rate of turbulent erosion. In § 3, the predictions of the two intermittency models are calculated. The data are briefly reviewed in § 4, and conclusions are drawn. Some additional predictions in the case of the β model are discussed in § 5.

2. Model for turbulent erosion of a stably stratified fluid

The turbulent flow is characterized by some integral (largest) length scale l_0 and by some r.m.s. velocity u_0 of the turbulent eddies on the integral scale. We assume that the Reynolds number *Re* of the integral scale is large, with

$$Re \gg 1 \tag{1}$$

strongly satisfied, so that the inertial range extends over some logarithmic interval to smaller scales. Also, we make the usual convenient simplification of considering a discrete sequence of eddy scales,

$$l_n = l_0 2^{-n}, \quad n = 0, 1, 2\ldots. \tag{2}$$

What we require of an intermittency model, for our present purpose, is that it tell us the probability of a region of size l_n containing an eddy with mean-square velocity between u^2 and $u^2 + du^2$, i.e. that the model define a probability density function $P(u^2, l_n)$, normalized so that

$$\int P(u^2, l_n)\, du^2 = 1 \quad \text{for all } n. \tag{3}$$

Much of the probability *may* be concentrated in a delta function at $u^2 = 0$, if the intermittency model posits a 'clean' separation of active and inactive regions on scale l_n.

Before proceeding, we need to consider what other dimensionless numbers besides *Re* can enter the problem (cf. Turner 1968). The most important will be the Richardson number *Ri* associated with the density discontinuity $\delta\rho$ at the boundary between the turbulent and stable regions. As time increases and the turbulence erodes farther into the stable region, this density discontinuity will in general increase. Following Phillips (1977) we define

$$Ri \equiv \frac{g\, \delta\rho\, l_0}{\rho_0\, u_0^2} \tag{4}$$

where g is the acceleration of gravity and ρ_0 is the mean fluid density in the mixed region. The rate of turbulent erosion will depend primarily on *Ri*.

Of lesser importance is the Peclet number *Pe* associated with the diffusivity κ of the diffusion process (whether heat conductivity or saline molecular diffusivity),

$$Pe \equiv l_0 u_0 / \kappa. \tag{5}$$

In fact, as Turner (1968) points out somewhat differently, there is a regime where the rate of turbulent erosion should be independent of *Pe*. The physical picture is that a heavy wisp is lifted out of the interface by a turbulent eddy, and quickly mixed down to the Kolmogorov microscale by turbulent mixing. If the diffusivity is large enough so that, on scales below the Kolmogorov microscale, the wisp can completely lose its identity into its surroundings before gravity can pull it back to the interface, then the precise diffusion rate will not matter. We can make this condition quantitative at this point in the context of the Kolmogorov (1941, hereafter cited as K41) theory, and below we will comment on the small modifications required by intermittency effects.

An undiffused wisp is drawn downward with an acceleration $g(\delta\rho/\rho_0)$. According to K41, the size of the microscale l_μ is $l_0/Re^{\frac{3}{4}}$, and the diffusion time across this scale is l_μ^2/κ. Therefore, the condition that the wisp acquire a downward velocity no larger than the integral scale turbulent velocity is

$$g\frac{\delta\rho}{\rho_0}\frac{l_\mu^2}{\kappa} < u_0 \tag{6a}$$

which can be written as

$$Ri\, Pe < Re^{\frac{3}{2}}. \tag{6b}$$

Alternatively, one might take the point of view that, once eroded from the stable layer, a wisp will always have available at least *one* eddy turnover time to diffuse into the surrounding fluid. In this case, the condition analogous to equation (6*a*) is

$$\frac{l_\mu^2}{\kappa} < \frac{l_0}{u_0} \tag{7a}$$

which can be written as

$$Pe < Re^{\frac{1}{2}}, \tag{7b}$$

which is more easily satisfied than (6*b*) for $Ri > 1$. Turner (1968) found different results for turbulent erosion when the diffusive process involved heat and salt, and these differences seem possible to understand in terms of condition (6*b*) or (7*b*) holding in the former, but not the latter, case, because of the latter's higher Pe (see § 4).

Below, in the context of specific intermittency models, we will find that there are some other conditions on the relative magnitudes of Ri, Pe, and Re which must be assumed. For typical experiments with heat conduction in water, Re will pose the most stringent limits, since it is difficult to achieve

$$Re > 10^3 \tag{8}$$

for laboratory flows. We discuss this further below.

Let us consider now an eddy of size l that finds itself just at the interface of the stably stratified region. Is it able to scour out a pocket of size $\sim l$ from the stable region? The answer depends on whether the Reynolds stress of the eddy exceeds the force necessary to lift the stable pocket through a height l, i.e. whether

$$\rho u_1^2 > g\,\delta\rho\, l \tag{9}$$

(cf. Phillips 1977, equation 6.7.2). Equivalently, up to a constant of order unity, condition (9) can be described as requiring that the local shear on the scale l exceed the critical value necessary to destabilize the density interface via Kelvin–Helmholtz instability. Using equation (4), (9) becomes

$$\frac{u_1^2}{u_0^2} > Ri\frac{l}{l_0}. \tag{10}$$

When scouring is possible, it should proceed at about the eddy turnover velocity u_1 of the scouring eddy. Since $P(u^2, l_n)\,du^2$ is the fraction of volume in which eddies of velocity u^2 to u^2+du^2 are actually present, it should also be roughly the fraction of area of the interface plane upon which such eddies are incident at any given time. Therefore, the mean erosion velocity of the interface due to the scale l is

$$u_{e,l} = \int_{u_0^2\, Ri\, l/l_0}^{\infty} uP(u^2, l)\,du^2, \tag{11}$$

where the lower limit comes from equation (10). And, integrating over the logarithmically independent scales l_n, the total erosion velocity, scaled to the integral scale velocity u_0, is

$$\frac{u_e}{u_0} = (\ln 2)^{-1}\int_0^{l_0}\frac{dl}{l}\int_{Ri\,l/l_0}^{\infty}\frac{u}{u_0}P\left(\frac{u^2}{u_0^2}, l\right)d\left(\frac{u^2}{u_0^2}\right). \tag{12}$$

The equality is understood to allow for an additional overall constant of order unity. The right-hand side of (12) is a function of the Richardson number only. This can be made explicit by defining

$$p \equiv \ln(l_0/l), \quad s \equiv u^2/u_0^2 \tag{13}$$

so,

$$\frac{u_e}{u_0} = (\ln 2)^{-1}\int_{p=0}^{\infty}dp\int_{Ri\exp(-p)}^{\infty} s^{\frac{1}{2}}P(s,p)\,ds. \tag{14}$$

We will now proceed to use functions $P(s,p)$ from the log-normal and β models to calculate u_e/u_0.

3. Predictions of the log-normal and β models

A recent review of intermittency models in turbulence is found in Rose & Sulem (1978). The log-normal model is described by Kolmogorov (1962), Obukhov (1962), Yaglom (1966), Gurvich & Yaglom (1967) and Mandelbrot (1972). For our purposes here, we can take the model to say that the variable

$$y \equiv \ln\left(\frac{u^3 l_0}{u_0^3 l}\right) \tag{15}$$

where u_1 is an eddy velocity on scale l, is normally distributed with a variance

$$\sigma^2 = A + \mu\ln\left(\frac{l_0}{l}\right) \equiv A + \mu p \tag{16}$$

and with a mean value which is determined by the Kármán–Howarth constraint, that the average value of u_1^3/l be independent of l. In equation (16), μ is a universal dimensionless constant, while A depends on the macroscopic flow.

Equations (15) and (16), and the Kármán–Howarth condition, imply a probability density function

$$P(y,p)\,dy = (2\pi\sigma^2)^{-\frac{1}{2}}\exp\left[-(y+q)^2/2\sigma^2\right]dy \tag{17}$$

with

$$q = \tfrac{1}{2}(\mu p + A). \tag{18}$$

(Note that one changes variables in P by the rule for probability densities,

$$P[f(s), g(p)]\,d[f(s)] = P(s,p)\,ds.)$$

If $\mu = 0$, then we recover K41 without intermittency. A typical value which is supposed to model observed intermittency adequately is

$$\mu \approx \tfrac{1}{2}$$

although there is some evidence (e.g. Nelkin 1981) that a value as small as 0·25 might be preferred.

Equation (11) now gives, defining $\gamma \equiv \ln Ri$,

$$\begin{aligned} \frac{u_{el}}{u_0} &= \left(\frac{l}{l_0}\right)^{\frac{1}{3}} \int_{\frac{3}{2}\gamma - \frac{1}{2}p}^{\infty} e^{\frac{1}{3}y} (2\pi\sigma^2)^{-\frac{1}{2}} \exp\left[-(y+q)^2/2\sigma^2\right] dy \\ &= \frac{1}{2}\left(\frac{l}{l_0}\right)^{\frac{1}{3}} \operatorname{erfc}\left[\frac{\frac{3}{2}\gamma - \frac{1}{2}p + q - \frac{1}{3}\sigma^2}{(2\sigma^2)^{\frac{1}{2}}}\right] \exp\left[-\left(\frac{q}{3} - \frac{\sigma^2}{18}\right)\right]. \end{aligned} \tag{19}$$

The integral over scales $\int dp$ (equation (14)) cannot be evaluated in closed form, but its asymptotic limits for both $Ri > 1$ ($\gamma \to \infty$) and $Ri < 1$ ($\gamma \to -\infty$) are evident by inspection, since the complimentary error function erfc is essentially a step function at zero argument. The result is

$$\begin{aligned} \frac{u_e}{u_0} &\propto 1 \quad \text{for} \quad Ri < 1 \\ &\propto Ri^{-(1+\frac{1}{3}\mu)/(1-\frac{1}{3}\mu)} \quad \text{for} \quad Ri > 1. \end{aligned} \tag{20}$$

The constants of proportionality, which involve μ, are always near unity. With $\mu = 0{\cdot}5$, equation (20) gives

$$\frac{u_e}{u_0} \propto Ri^{-1{\cdot}40} \tag{21}$$

as a prediction of the model. Note that if $\mu = 0$, so that K41 is reproduced, the power-law dependence goes over to Ri^{-1}.

We now turn our attention to the β-model of Frisch *et al.* (1978), also reviewed by Rose & Sulem (1978), and based in part on earlier work of Mandelbrot (1976) and Novikov & Stewart (1964). Here the basic idea is that on scales $l_n < l_0$ the eddies of that scale are not space-filling, but are rather concentrated in 'active regions' that occupy a fraction β_n of the total volume. The filling factor β_n is taken to have a universal power-law dependence

$$\beta_n = (l_n/l_0)^{3-D} \tag{22}$$

where D is between 2 and 3, and is the 'self-similarity' or 'fractal' dimension. When the rate of energy transfer between scales n and $n+1$ is taken to be independent of n, as in K41, one obtains a dependence of eddy velocity u_l on l,

$$u_l/u_0 = (l/l_0)^{\frac{1}{3}(D-2)}. \tag{23}$$

K41 is obtained in the limiting case of $D = 3$; the best value of D to explain intermittency data is supposed to be 2·5 or possibly 2·75.

In its simplest form, the model gives only a single characteristic velocity for u_l, rather than a probability distribution, so our probability function $P(s,p)$ will have the form

$$P(s,p) = \beta_n \delta(s - s_n) + (1 - \beta_n)\delta(s) \tag{24}$$

where the δ's are Dirac delta functions. The integral of (11) gives

$$\begin{aligned} \frac{u_e}{u_0} &= \left(\frac{l}{l_0}\right)^{\frac{1}{3}(D-2)} \left(\frac{l}{l_0}\right)^{3-D} \quad \text{for} \quad \left(\frac{l}{l_0}\right)^{\frac{2}{3}(D-2)} \geqslant Ri\left(\frac{l}{l_0}\right) \\ &= 0 \quad \text{for} \quad \left(\frac{l}{l_0}\right)^{\frac{2}{3}(D-2)} \leqslant Ri\left(\frac{l}{l_0}\right). \end{aligned} \tag{25}$$

Equation (12) then gives

$$\frac{u_e}{u_0} = (\ln 2)^{-1} \int_0^{l/l_0 = Ri^{3/(2D-7)}} \left(\frac{l}{l_0}\right)^{\frac{1}{3}(7-2D)} \frac{dl}{l} = \frac{3}{(7-2D)\ln 2} Ri^{-1}. \tag{26}$$

The somewhat surprising result is that the β-model gives an Ri^{-1} dependence *independent of D*, by contrast with the log-normal model which gives this dependence only in the K41 limit. It is not difficut to show (Nelkin, private communication) that Ri^{-1} is also obtained in the case of a generalized β model which allows an arbitrary, but self-similar, distribution of velocities in the active regions (instead of the single characteristic velocity assumed above).

We are now in a position to note the limits of validity of this Ri^{-1} dependence which were alluded to in the discussion preceding equation (8). The largest eddy which is able to erode has (equation (25))

$$\frac{l}{l_0} = Ri^{3/(2D-7)}, \tag{27}$$

while the Kolmogorov microscale has a size of order

$$\frac{l_\mu}{l_0} = Re^{-3/(10-2D)} \tag{28}$$

(cf. Rose & Sulem 1978, equations 7.4–7.6). The conditions that (27) be larger than (28), so that u_e/u_0 should be almost independent of *Re*, is $Ri^4 < Re$ for $D = 3$, which is not well satisfied by laboratory experiments; for $D = 2{\cdot}5$, however, it is

$$Ri^{\frac{5}{2}} < Re \tag{29}$$

which is marginally satisfied, at least for *Ri* not too large.

The conditions which yields (6*b*) or (7*b*) in the K41 theory are slightly modified when $D = 2{\cdot}5$ and become

$$Ri\, Pe < Re^{\frac{6}{5}} \tag{30a}$$

or

$$Pe < Re^{\frac{6}{5}}. \tag{30b}$$

Finally, for the above model to hold, one ought to require that the diffusion thickness of the interface, as it is eaten away at velocity u_e, is smaller than the size of the eroding eddy. The diffusion thickness is given by

$$l_d \sim \frac{\kappa}{u_e} \sim \frac{\kappa Ri}{u_0} \tag{31}$$

so, comparing to equations (4), (5), and (27),

$$\frac{Ri}{Pe} < Ri^{3/(2D-7)}. \tag{32}$$

For $D = 2{\cdot}5$ this is

$$Ri^{\frac{5}{2}} < Pe. \tag{33}$$

This condition is more restrictive than a related one, derived by Phillips (1977, equation 6.7.3) in a different context, that the turbulent erosion proceed faster than the stable stratification is restored by the diffusion process. Phillips' condition gives,

$$Ri^2 < Pe$$

independent of *D*.

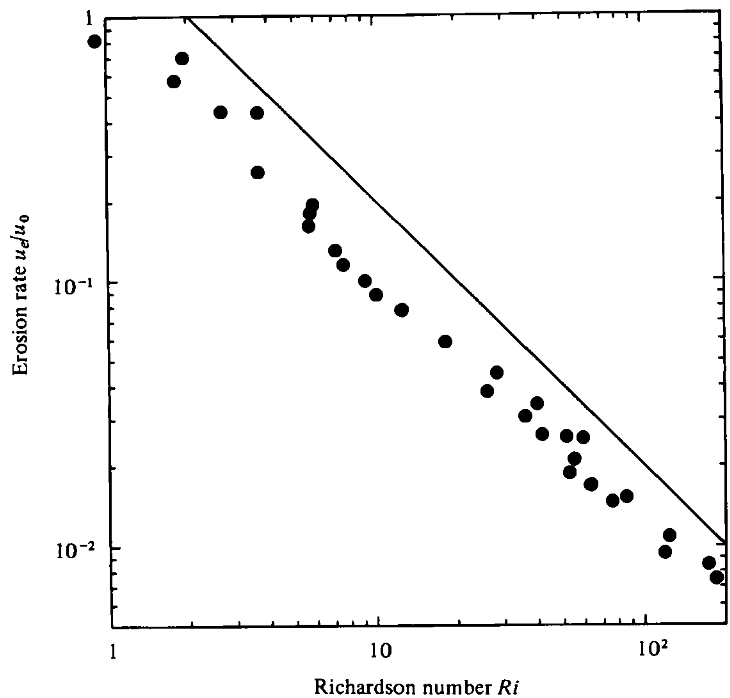

FIGURE 1. Erosion rates in a fluid stratified by a temperature gradient, plotted against Richardson number of the integral turbulent scale and the density discontinuity, from Turner (1968, 1973) and Thompson (1969). The solid line is the Ri^{-1} prediction of the β-model, normalized by equation (26) with $D = 2{\cdot}5$ (i.e. *not* adjusted to fit the data). The Kolmogorov log-normal model predicts a steeper slope, $Ri^{-1{\cdot}4}$.

4. Review of data and conclusions

Turner (1973, ch. 9) and Phillips (1977) review the experimental data on mixing across density interfaces in some detail, including the works of Rouse & Dodu (1955), Cromwell (1960), Turner (1968), Thompson (1969), Kato & Phillips (1969), Kantha & Phillips (1977), and others. In the interpretation of these results, one must bear in mind the variety of conditions represented, not only in terms of the parameter space of *Re*, *Pe*, *Ri*, but also in terms of issues such as the presence or absence of a mean shear in the turbulent region, relative to the stable layer. (Our model assumes no such mean shear.)

The experiment of Turner (1968) on a temperature stratified fluid, with flow parameters measured by Thompson (1969), seems to have the most direct bearing on the predictions of this paper, since conditions (1), (29), (30), and (33) are satisfied, or violated only weakly (at the higher values of *Ri*). Typical parameters for the experiment are $Re \sim 10^3$, $Pe/Re\ (= Pr) \sim 10$, and $Ri \sim 10$–100. Published data from this experiment are shown in figure 1.

It is immediately evident that the data vary as Ri^{-1}, which is the prediction of equation (26) for the β-model, and that the data seem to be incompatible with the prediction of the log-normal model for $\mu = \frac{1}{2}$, namely $Ri^{-1{\cdot}4}$ (equation (21)). The physical root of the difficulty with the log-normal model is that its high-velocity tail at small scales is much too powerful, so that the Kármán–Howarth condition forces down the mean eddy velocity, and thus the model predicts too fast a decrease of erosion with increasing Ri.

The solid line in figure 1 has slope Ri^{-1}, and has the normalization of equation (26), without any correction factor. In fact, a correction factor ought to be allowed, since our fundamental assumptions, that a scouring eddy erodes at its eddy velocity u_l, and that eddies separated by a factor 2 in l contribute independently, can certainly be true only up to a factor of order unity. If one fits to the data, one finds that a factor of about $\frac{1}{2}$ should be inserted on the right-hand side of equation (11) and subsequent equations.

Unfortunately, there are several reasons for skepticism in regarding the existing experimental evidence as a definitive test of the intermittency models. First, the required inequalities for validity of the model are not strongly satisfied by existing experiments. The fundamental problem is that Re is too small. In fact, at $Re = 10^3$ one has only barely begun to have a recognizable inertial range in the turbulent cascade. A related point is that there may be a minimum value of Re for which departures from K41 become observable. Van Atta & Antonia (1980) suggest $Re > 3 \times 10^3$, which is larger than achieved in Turner's experiments.

Second, the values of μ is uncertain, and might be as small as 0·25 (Nelkin 1980). As μ approaches zero, the predictions of the log-normal model become identical to those of the K41 limit. For these reasons, the striking agreement with the data may, at the presently accessible experimental range of Re, only reflect agreement with the limiting case of K41. Experiments at larger Re could be more definitive.

Turner's (1968, 1973) data on the salt-stratified case does lend some support to the present model, at least indirectly. Those data have the same Ri and Re, but Pe is a factor of 10^2 larger. This causes conditions (30a) and (30b) to fail rather drastically for $Ri > 3$. One in fact sees in the data that the salt and heat points are compatible up to about this value, while for larger values of Ri, the salt data fall on a different power law (estimated by Turner as $Ri^{-\frac{3}{2}}$).

Our conclusion, which given the simplicity of the model and relative paucity of the data must naturally be tentative, is that the β-model is in good agreement with the data, predicting in fact the Ri^{-1} behaviour that it is observed. If Ri^{-1} persists at higher values of Re, then it will conflict with the prediction of the log-normal model.

5. Possibility of further tests

The Ri^{-1} prediction of the β-model is independent of the value of its unknown fractal dimension D ($2 < D < 3$). On the other hand, the region of validity of this prediction does depend on D, namely as

$$Ri\, Pe < Re^{3/(5-D)} \tag{34a}$$

(generalization of equations (6b) and (30a), or

$$Pe < Re^{3/(5-D)} \tag{34b}$$

(generalization of equations (7b) and (30b);

$$Ri^{(10-2D)/(7-2D)} < Pe \tag{35}$$

(from equation (32)); and

$$Ri^{(10-2D)/(7-2D)} < Re \tag{36}$$

(from equations (27) and (28)). All three of these conditions are inequalities with Ri on the small side. This suggests that one might determine D (and simultaneously check the model of this paper) by measuring where the *break* in the Ri^{-1} law occurs as Ri is increased, and determining the functional variation of this break with changing Pe and Re.

One needs to keep in mind that the inequalities are only approximate, and they may have unknown constants of order one on their right-hand sides; however the value of D can, in principle, be determined independently of these constants, since D affects the power-law slope of the variation of the break with Pe or Re.

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